The envelope theory and the improved envelope theory: an overview of these approximation methods

Presented by Cyrille Chevalier and Lorenzo Cimino

September 8, 2022

Nuclear and subnuclear physics unit
University of Mons

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The ET for systems of all identical particles

What is the ET ?

The envelope theory (ET) is an approximation method to solve the many-body Schrödinger equation.

[1] Semay, Ducobu (2016) Eur. J. Phys., **37**, 045403

The ET for systems of all identical particles

What is the ET ?

The envelope theory (ET) is an approximation method to solve the many-body Schrödinger equation.

It provides approximation of the spectrum and eigenvectors for a very large class of N-body Hamiltonians [1].

[1] Semay, Ducobu (2016) Eur. J. Phys., **37**, 045403

The ET for systems of all identical particles

What is the ET ?

For today, we will use:

$$H = \sum_{i=1}^{N} T(p_i) + \sum_{i<j=2}^{N} V(r_{ij})$$

with $p_i = |\vec{p}_i|$ and $r_{ij} = |\vec{r}_i - \vec{r}_j|$.

One- and K-body potentials can also be considered. All computations are performed in the centre of mass (CM) frame.

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[2] Silvestre-Brac, Semay, Buisseret (2012) J. Phys. Math., **4**, 120601

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One- and K-body potentials can also be considered. All computations are performed in the centre of mass (CM) frame.

The basic idea of the ET is to approximate this hamiltonian with a set of **harmonic oscillator (HO) Hamiltonian** [2].

[2] Silvestre-Brac, Semay, Buisseret (2012) J. Phys. Math., **4**, 120601

The ET for systems of all identical particles

What is the ET ?

Remark about the N identical body HO:

$$H_{OH} = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<j=2}^{N} \rho \vec{r}_{ij}^{\,2} - \frac{\vec{P}^2}{2Nm} \quad \text{with } \vec{P} = \sum_{i=1}^{N} \vec{p}_i.$$

[3] Silvestre-Brac, Semay, Buisseret, Brau (2010) J. Math. Phys., **51**, 032104

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The exact spectrum for this Hamiltonian can be analytically found by a diagonalisation procedure [3]:

$$E_{\text{HO}} = \sqrt{\frac{2N\rho}{m}} Q(N) \text{ with } Q(N) = \sum_{i=1}^{N-1} (2n_i + l_i + D/2)$$

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(We use natural units).

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The ET for systems of all identical particles

Essential idea behind the ET

Aim: find the spectrum of this generic Hamiltonian

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To this end, we introduce an **auxiliary Hamiltonian**:

$$\tilde{H} = \sum_{i=1}^{N} \tilde{T}_i(p_i, \{\mu_i\}) + \sum_{i<j=2}^{N} \tilde{V}_{ij}(r_{ij}, \{\rho_{ij}\}).$$

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- $\{\mu_i\}$ and $\{\rho_{ij}\}$ are called **auxiliary fields** [2]. For now, they depend on the variables $\vec{p}_i$ and $\vec{r}_i$.

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Commentaries about the auxiliary Hamiltonian:

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- Illustration for the potential:

$$\tilde{V}_{ij}(r_{ij}, \{\rho_{ij}\}) = \rho_{ij} r_{ij}^2 + V(J(\rho_{ij})) + \rho_{ij} J(\rho_{ij})^2$$

where $J(x)$ is the inverse of $V'(x)/(2x)$.

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where $J(x)$ is the inverse of $V'(x)/(2x)$.

- Setting the constraints $\frac{\delta \tilde{H}}{\delta \mu_i} = \frac{\delta \tilde{H}}{\delta \rho_{ij}} = 0 \;\; \forall i, j,\; H$ is recovered [2].

  $\Rightarrow$ Auxiliary field method

[2] Silvestre-Brac, Semay, Buisseret (2012) J. Phys. Math., **4**, 120601

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Essential idea behind the ET

Idea behind the ET: to replace the **auxiliary fields** by **auxiliary parameters** [2]. $\tilde{H}$ becomes then an HO Hamiltonian.

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- $\tilde{V}_{ij}(r_{ij}, \{\rho_{ij}\}) = \rho_{ij} r_{ij}^2 + \underbrace{V(J(\rho_{ij})) + \rho_{ij} J(\rho_{ij})^2}_{C^{st}}$.

  $\Rightarrow$ $\tilde{E} = E_{\text{HO}}(\{\mu_i\}, \{\rho_{ij}\}) + C^{st}(\{\mu_i\}, \{\rho_{ij}\})$, where $\tilde{E}$ is an eigenvalue of $\tilde{H}$.

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  $\Rightarrow$ $\tilde{E} = E_{\text{HO}}(\{\mu_i\},\{\rho_{ij}\}) + C^{st}(\{\mu_i\},\{\rho_{ij}\})$, where $\tilde{E}$ is an eigenvalue of $\tilde{H}$.

- The spectrum of H is approximately recovered by setting [2]

$$\left.\frac{\partial \tilde{E}}{\partial \mu_i}\right|_{\mu_{i0}} = \left.\frac{\partial \tilde{E}}{\partial \rho_{ij}}\right|_{\rho_{ij0}} = 0 \quad \forall i,j.$$

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Essential idea behind the ET

How many equations do we need to solve ?

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- In general, there are N parameters μ_i and $C_N^2 = \frac{N(N-1)}{2}$ parameters ρ_{ij}.

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- In general, there are N parameters μ_i and $C_N^2 = \frac{N(N-1)}{2}$ parameters ρ_{ij}.
- Since the wave-function must be (anti-)symmetric, we can show $\mu_i = \mu$ and $\rho_{ij} = \rho \ \ \forall i, j$.

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- Since the wave-function must be (anti-)symmetric, we can show $\mu_i = \mu$ and $\rho_{ij} = \rho \quad \forall i, j$.
- The constraints $\frac{\partial \tilde{E}}{\partial \mu_i} = \frac{\partial \tilde{E}}{\partial \rho_{ij}} = 0$ reduces to only two equations and to two solutions (μ_0, ρ_0) for each level.

The ET for systems of all identical particles

Essential idea behind the ET

⇒ Basically ***H*** **is approximated by a set of auxiliary HO Hamiltonians**, one for each level [2].

[2] Silvestre-Brac, Semay, Buisseret (2012) J. Phys. Math., **4**, 120601

[4] Semay, Roland (2013) Res. Phys., **3**, 231

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Essential idea behind the ET

$\Rightarrow$ Basically ***H* is approximated by a set of auxiliary HO Hamiltonians**, one for each level [2].

It can be shown that the constraints

$$\left.\frac{\partial \tilde{E}}{\partial \mu}\right|_{\mu_0} = \left.\frac{\partial \tilde{E}}{\partial \rho}\right|_{\rho_0} = 0$$

lead to a set of three equations called **compact equations of the ET** [4].

[2] Silvestre-Brac, Semay, Buisseret (2012) J. Phys. Math., **4**, 120601

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The ET for systems of all identical particles

Compact equations

Practical user necessary informations: let us take the following generic Hamiltonian

$$H = \sum_{i=1}^{N} T(|\vec{p}_i|) + \sum_{i<j=2}^{N} V(|\vec{r}_i - \vec{r}_j|),$$

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$$H = \sum_{i=1}^{N} T(|\vec{p}_i|) + \sum_{i<j=2}^{N} V(|\vec{r}_i - \vec{r}_j|),$$

the next system gives an approximation for its spectrum [4]:

$$\begin{cases} \tilde{E} = NT(p_0) + C_N^2 V(\rho_0) \\ Np_0\, T'(p_0) = C_N^2 \rho_0\, V'(\rho_0) \\ \sqrt{C_N^2}\rho_0\, p_0 = Q(N) \end{cases}$$

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- with $p_0^2 = \langle \vec{p}_i^{\,2} \rangle$ and $\rho_0^2 = \langle (\vec{r}_i - \vec{r}_j)^2 \rangle \quad \forall i, j$,

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- with $p_0^2 = \langle \vec{p}_i^{\,2} \rangle$ and $\rho_0^2 = \langle (\vec{r}_i - \vec{r}_j)^2 \rangle \quad \forall i, j,$
- with $Q(N) = \sum_{i=1}^{N-1} (2n_i + l_i) + (N-1)\frac{D}{2}$.

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Compact equations

Parenthesis about the derivation of the compact equations [4] :

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Parenthesis about the derivation of the compact equations [4] :

- Hellmann-Feynman theorem: $\left\langle \frac{\partial H}{\partial \alpha} \right\rangle = \frac{\partial E}{\partial \alpha}$
  $\Rightarrow \; \tilde{E} = N T(p_0) + C_N^2 V(\rho_0)$

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- (Generalised) virial theorem
  $\Rightarrow N p_0 T'(p_0) = C_N^2 \rho_0 V'(\rho_0)$
- Comparison with an exact solution: $\langle H_{\text{HO}} \rangle = E_{\text{HO}}$
  $\Rightarrow \sqrt{C_N^2} \rho_0 p_0 = Q(N)$

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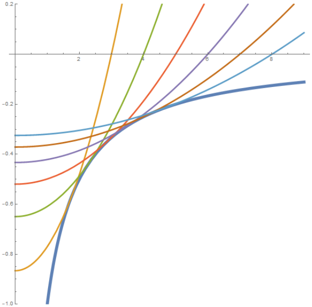
The ET for systems of all identical particles

Properties of the ET

- **Variational character**: ET can give an upper or a lower bound [2].



[2] Silvestre-Brac, Semay, Buisseret (2012) J. Phys. Math., **4**, 120601
[4] Semay, Roland (2013) Res. Phys., **3**, 231
[5] Semay (2015) Few-Body Syst., **56**, 149

The ET for systems of all identical particles

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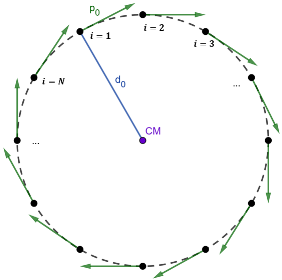
- The compact equations have a nice **semi-classical interpretation** [4].



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Properties of the ET

- **Variational character**: ET can give an upper or a lower bound [2].
- The compact equations have a nice **semi-classical interpretation** [4].
- Solution may be **analytical** with N as a variable [5].

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A variational character for ET

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The ET for systems of all identical particles

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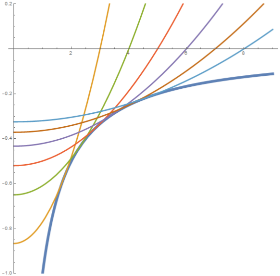

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A variational character for ET

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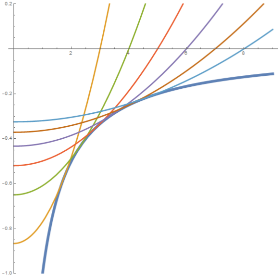

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The ET for systems of all identical particles

Analycity

Hamiltonian: $T(p) = \frac{\vec{p}\,^2}{2m}$ $V(x) = ax^2$

$\Rightarrow$ "Approximated" spectrum: $E = Q\sqrt{\frac{2N}{m}a}$

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

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Hamiltonian: $T(x) = Fx^{\alpha}$ $V(x) = \mathrm{sgn}(\beta)Gx^{\beta}$

$\Rightarrow$ Approximated spectrum:

$$E = \mathrm{sgn}(\beta)(\beta+\alpha)\left(\left(\frac{NF}{|\beta|}\right)^{\beta}\left(\frac{G}{\alpha}\right)^{\alpha}\left(\sqrt{C_N^2}\right)^{2\alpha-\alpha\beta} Q^{\alpha\beta}\right)^{1/(\alpha+\beta)}$$

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Hamiltonian: $T(p) = \frac{\vec{p}^{\,2}}{2m}$ $V(x) = -V_g e^{-\frac{(x_i-x_j)^2}{a^2}}$

$\Rightarrow$ Approximated spectrum : $E = -C_N^2 V_g e^{2W(\delta)}(2W(\delta)+1)$ where $\delta = -\frac{1}{2}\left(\frac{N}{V_g 2m (C_N^2)^2 a^2} Q^2\right)^{1/2}$ and $W(x)$ is a Lambert function.

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

The ET for systems of all identical particles

Tests at $D = 3$

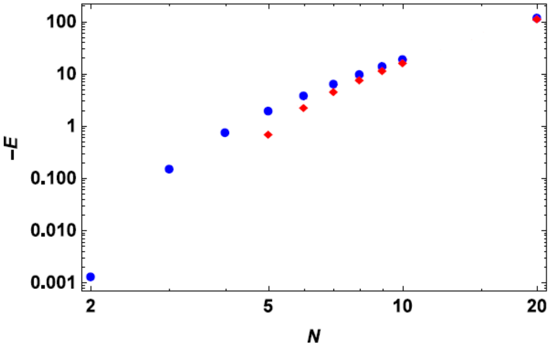

Figure: Biding energy for weakly-interacting bosons (gaussian interaction) - Exact results in circles, ET results in diamonds.

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

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Tests at $D = 3$

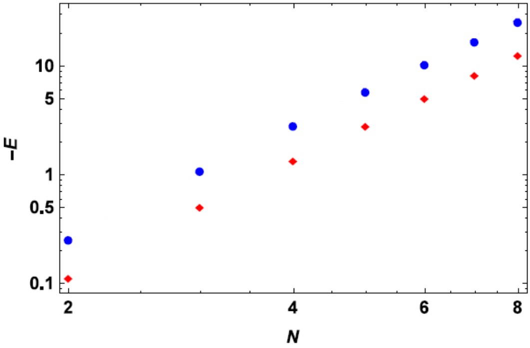

Figure: Biding energy for self-gravitating bosons (coulomb interaction) - Exact results in circles, ET results in diamonds.

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Test at $D = 1$

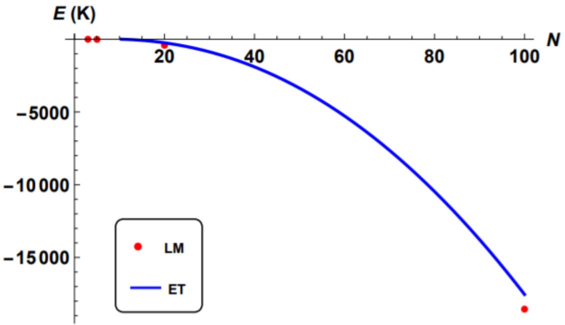

Figure: Biding energy for weakly-interacting bosons (gaussian interaction) - Exact results in circles, ET results in line.

Results are from [7] Semay, Cimino (2019) Few-Body Syst., **60**, 64

The ET for systems of $N_a + 1$ particles

Why to generalize ?

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Hybrid baryons: three quarks + one constituent gluon

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- Constituent approach:
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Hybrid baryons: three quarks + one constituent gluon

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- Large-N approach of QCD :
  - N quarks + one gluon

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Hybrid baryons: three quarks + one constituent gluon

- Constituent approach:
  - Interaction with gluonic field $\rightsquigarrow$ potential
- Large-N approach of QCD :
  - N quarks + one gluon
- Combination:
  - To solve a $N_a + 1$ particles quantum system

The ET for systems of $N_a + 1$ particles

Compact equations

Remark about the HO: for a system of $N_a + 1$ particles, the HO reads as

$$H_{\text{HO}} = \sum_{i=1}^{N_a} \frac{\vec{p}_i^{\,2}}{2\mu_a} + \frac{\vec{p}_b^{\,2}}{2\mu_b} + \sum_{i<j=2}^{N_a} \rho_{aa}(|\vec{r}_i - \vec{r}_j|)^2 + \sum_{i=1}^{N_a} \rho_{ab}(|\vec{r}_i - \vec{r}_b|)^2.$$

[9] Semay, Cimino, Willemyns (2020) Few-Body Syst., **61**, 19

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After some manipulations [9], we can decompose the Hamiltonian as $H_{\text{HO}} = H_a + H_{\text{CM}}$ where

[9] Semay, Cimino, Willemyns (2020) Few-Body Syst., **61**, 19

The ET for systems of $N_a + 1$ particles

Compact equations

Remark about the HO: for a system of $N_a + 1$ particles, the HO reads as

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  $\Rightarrow$ HO of N_a identical particles

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  $\Rightarrow$ HO for the relative motion between the the CM of the N_a particles, and the different one

[9] Semay, Cimino, Willemyns (2020) Few-Body Syst., **61**, 19

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[8] Cimino, Semay (2022) Braz. J. Phys, **52**, 45

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the next system gives an approximation for its spectrum [8]:

$$\begin{cases} E = N_a T_a(p'_a) + T_b(P_0) + C^2_{N_a} V_{aa}(r_{aa}) + N_a V_{ab}(r'_0) \\ N_a T'_a(p'_a) \frac{p_a^2}{p'_a} = C_{N_a} V'_{aa}(r_{aa}) r_{aa} + \frac{N_a - 1}{2} V'_{ab}(r'_0) \frac{r_{aa}^2}{r'_0} \\ T'_b(P_0) P_0 + \frac{1}{N_a} T'_a(p'_a) \frac{P_0^2}{p'_a} = N_a V'_{ab}(r'_0) \frac{R_0^2}{r'_0} \\ p_a r_{aa} \sqrt{C^2_{N_a}} = Q(N_a) \\ P_0 R_0 = Q(2) \end{cases}$$

- with $p'^{\,2}_a = p_a^2 + \frac{p_0^2}{N_a^2}$ and $r'^{\,2}_0 = \frac{N_a - 1}{2N_a} r_{aa}^2 + R_0^2$
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[8] Cimino, Semay (2022) Braz. J. Phys, **52**, 45

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The ET for systems of $N_a + 1$ particles

Compact equations

Comparison with compact equations for systems of all identical particles:

$$\begin{cases} E = NT(p_0) + C_N^2 V(\rho_0) \\ Np_0 T'(p_0) = C_N^2 \rho_0 V'(\rho_0) \\ \sqrt{C_N^2}\rho_0 p_0 = Q \end{cases} \longleftrightarrow \begin{cases} E = N_a T_a(p_a') + T_b(P_0) + C_{N_a}^2 V_{aa}(r_{aa}) + N_a V_{ab}(r_0') \\ N_a T_a'(p_a') \frac{p_a^2}{p_a'} = C_{N_a} V_{aa}'(r_{aa}) r_{aa} + \frac{N_a - 1}{2} V_{ab}'(r_0') \frac{r_{aa}^2}{r_0'} \\ T_b'(P_0) P_0 + \frac{1}{N_a} T_a'(p_a') \frac{P_0^2}{p_a'} = N_a V_{ab}'(r_0') \frac{R_0^2}{r_0'} \\ p_a r_{aa} \sqrt{C_{N_a}^2} = Q(N_a) \\ P_0 R_0 = Q(2) \end{cases}$$

The ET for systems of N_a+1 particles

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---

[8] Cimino, Semay (2022) Braz. J. Phys, **52**, 45

The ET for systems of $N_a + 1$ particles

Tests

Test: $H = \sum_{i=1}^{3} |\vec{p}_i| + (\vec{r}_1 - \vec{r}_2)^2 + \kappa \sum_{i=1}^{2} (\vec{r}_i - \vec{r}_3)^2 \quad (D = 3)$

Results are from [9] Semay, Cimino, Willemyns (2020) Few-Body Syst., **61**, 19

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- $\kappa = 0.1, 10$:

(n_a, n_b, l_a, l_b)	κ	Exact [8,9]	ET	$\Delta(\%)$
$(0,0,0,0)$	0.1	5.288	5.597	5.5
	10	14.506	15.352	5.8
$(0,0,1,1)$	0.1	7.515	7.868	4.7
	10	20.340	21.580	6.1
$(1,0,0,0)$	0.1	8.067	8.570	6.2
	10	19.134	20.272	5.9
$(0,1,0,0)$	0.1	6.750	6.970	3.2
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The ET for systems of all identical particles

To improve the ET

- **The improved ET**: Q has a strong degeneracy.

[7] Lobashev, Trunov (2009) J. Phys. A, **42**, 345202
[5] Semay (2015) Few-Body Syst., **56**, 149
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  - For $\phi = 2$, we recover Q

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    - **Dominantly Orbital State Method** [10]

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The ~~envelope theory and the~~ improved envelope theory: an overview of these approximation methods

Presented by Cyrille Chevalier ~~and Lorenzo Cimino~~

September 8, 2022

Nuclear and subnuclear physics unit
University of Mons

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The DOSM for systems of all identical particles

Recap of the necessary informations about ET

Compact equations of ET (N identical particles): [1]

$$\begin{cases} E = NT(p_0) + C_N^2 V(\rho_0) \\ N p_0 T'(p_0) = C_N^2 \rho_0 V'(\rho_0) \\ \sqrt{C_N^2} \rho_0 p_0 = Q \end{cases}$$

[1] Semay, Roland (2013) Res. Phys., **3**, 231

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To improve the accuracy of the ET: [2]

$$Q \to Q_\phi = \phi \sum_{i=1}^{N-1}\left(n_i + \frac{1}{2}\right) + \lambda(l_i)$$

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The ET for systems of all identical particles

A semi-classical interpretation

Semi-classical interpretation for compact equations [1]:

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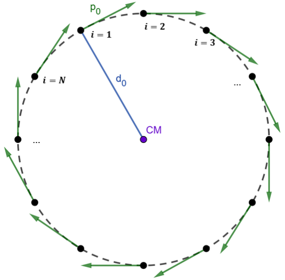

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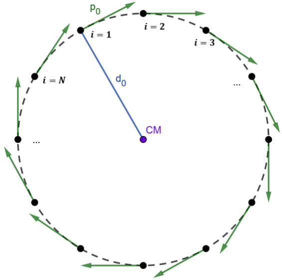

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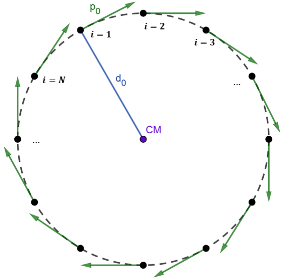

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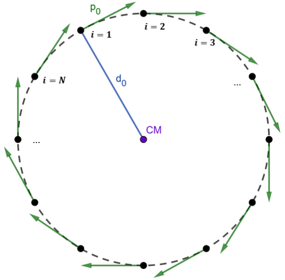

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What is the DOSM ?

Strategy behind the DOSM [2]:

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[2] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

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Strategy behind the DOSM [2]:

(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

[2] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

What is the DOSM ?

Strategy behind the DOSM [2]:

(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

(3) To quantify the perturbation

[2] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

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The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

(1) To start with a classical purely orbital solution

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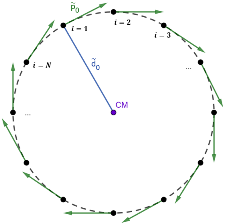
$$\begin{cases} \tilde{E}_0 = NT(\tilde{p}_0) + C_N^2 V(\tilde{\rho}_0) \\ \sqrt{C_N^2}\tilde{\rho}_0\tilde{p}_0 = \lambda \\ N\tilde{p}_0 T'(\tilde{p}_0) = C_N^2\tilde{\rho}_0 V'(\tilde{\rho}_0) \end{cases}$$



The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

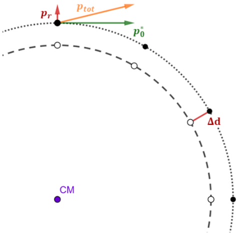
(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

(3) To quantify the perturbation

$$\tilde{\rho}_0 \xrightarrow{pert.} \tilde{\rho}_0 + \Delta\rho$$

$$\tilde{p}_0 \xrightarrow{pert.} \sqrt{p_r^2 + \left(\frac{\lambda}{\sqrt{C_N^2(\tilde{\rho}_0 + \Delta\rho)}}\right)^2}$$



The DOSM for systems of all identical particles

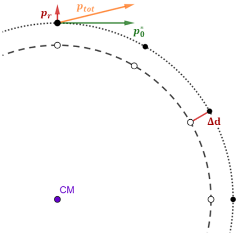
To apply the DOSM to the compact equations

Strategy:

(1) To start with a classical purely orbital solution
(2) To start a radial perturbation (still classically)
(3) To quantify the perturbation

$$\tilde{\rho}_0 \xrightarrow{pert.} \tilde{\rho}_0 + \Delta\rho$$

$$\tilde{p}_0 \xrightarrow{pert.} \tilde{p}_0 \left(1 + \frac{p_r^2}{2\tilde{p}_0{}^2} - \frac{\Delta\rho}{\tilde{\rho}_0} + \frac{\Delta\rho^2}{\tilde{\rho}_0{}^2}\right)$$



The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

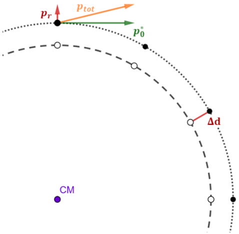
(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

(3) To quantify the perturbation

$$\tilde{E}_0 = N T(\tilde{p}_0) + C_N^2 V(\tilde{\rho}_0) \xrightarrow{pert.}$$

$$\Delta E = \left(\frac{N}{2\tilde{p}_0} T'(\tilde{p}_0) \right) p_r^2 + \left(\frac{N\tilde{p}_0}{\tilde{\rho}_0{}^2} T'(\tilde{p}_0) + \frac{N\tilde{p}_0{}^2}{2\tilde{\rho}_0{}^2} T''(\tilde{p}_0) + \frac{C_N^2}{2} V''(\tilde{\rho}_0) \right) \Delta\rho^2$$



The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

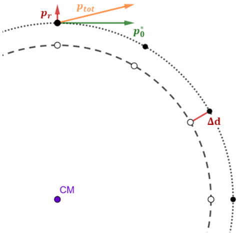
(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

(3) To quantify the perturbation

$$\tilde{E}_0 = NT(\tilde{\rho}_0) + C_N^2 V(\tilde{\rho}_0) \xrightarrow{pert.}$$

$$\Delta E = \frac{p_r^2}{2\mu} + \frac{k}{2}\Delta\rho^2$$



The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

(3) To quantify the perturbation

$$\Delta E = \frac{p_r^2}{2\mu} + \frac{k}{2}\Delta\rho^2 \;\rightarrow\; \Delta E = \sqrt{\frac{k}{\mu}}\left(n+\frac{1}{2}\right)$$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

(1) To start with a classical purely orbital solution

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The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

(3) To quantify the perturbation

Test on HO $\rightarrow \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right) = \sqrt{C_N^2} \left(n + \frac{1}{2} \right)$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Aim: to compare ET and DOSM in the same conditions.

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Aim: to compare ET and DOSM in the same conditions.

(1) To restructure Q_ϕ,

$$Q_\phi = \phi \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right) + \lambda$$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Aim: to compare ET and DOSM in the same conditions.

(1) To restructure Q_ϕ,

$$Q_\phi = \phi \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right) + \lambda$$

$$= \lambda\epsilon + \lambda \qquad \text{avec } \epsilon = \frac{\phi}{\lambda} \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right)$$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Aim: to compare ET and DOSM in the same conditions.

(1) To restructure Q_ϕ,

(2) To develop for $\epsilon \ll 1$ (radial perturbation),

$$Q_\phi = \lambda\epsilon + \lambda$$

$$\rho_0 = \tilde{\rho}_0 + \Delta\rho$$

$$p_0 = \tilde{p}_0 + \Delta p$$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Aim: to compare ET and DOSM in the same conditions.

(1) To restructure Q_ϕ,

(2) To develop for $\epsilon \ll 1$ (radial perturbation),

$$\tilde{E}_0 = NT(\tilde{p}_0) + C_N^2 V(\tilde{\rho}_0) \xrightarrow{pert.}$$

$$\Delta E = N\tilde{p}_0 T'(\tilde{p}_0)\epsilon$$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Aim: to compare ET and DOSM in the same conditions.

(1) To restructure Q_ϕ,

(2) To develop for $\epsilon \ll 1$ (radial perturbation),

$$\tilde{E}_0 = NT(\tilde{p}_0) + C_N^2 V(\tilde{\rho}_0) \xrightarrow{pert.}$$

$$\Delta E = N\tilde{p}_0\, T'(\tilde{p}_0)\frac{\phi}{\lambda}\sum_{i=1}^{N-1}\left(n_i + \frac{1}{2}\right)$$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

(3) Determination of ϕ :

$$\text{DOSM} \to \Delta E = \sqrt{\frac{k}{C_N^2 \mu}} \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right),$$

$$\text{improved ET} \to \Delta E = N \tilde{p}_0 T'(\tilde{p}_0) \frac{\phi}{\lambda} \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right).$$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

(3) Determination of ϕ :

$$\phi = \frac{\lambda}{N\tilde{\rho}_0 T'(\tilde{\rho}_0)} \sqrt{\frac{k}{C_N^2 \mu}}$$

$$\text{with } \mu = \frac{\tilde{\rho}_0}{NT'(\tilde{\rho}_0)}$$

$$\text{and } k = \frac{2N\tilde{\rho}_0}{\tilde{\rho}_0{}^2} T'(\tilde{\rho}_0) + \frac{N\tilde{\rho}_0{}^2}{\tilde{\rho}_0{}^2} T''(\tilde{\rho}_0) + C_N^2 V''(\tilde{\rho}_0)$$

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The DOSM for systems of all identical particles

Methodology

Methodology [2]:

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[2] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

Methodology

Methodology [2]:

- To choose l_i and calculate λ,

$$\boldsymbol{\lambda} = \sum_{i=1}^{N-1} l_i + (N-1)\frac{D-2}{2}$$

[2] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

Methodology

Methodology [2]:

- To choose l_i and calculate λ,
- To solve the ET compact equations for a purely orbital excitation (to find $\tilde{\rho}_0$ and $\tilde{p}_0$),

$$\begin{cases} \sqrt{C_N^2}\tilde{\rho}_0\tilde{p}_0 = \lambda \\ N\tilde{p}_0 T'(\tilde{p}_0) = C_N^2\tilde{\rho}_0 V'(\tilde{\rho}_0) \end{cases}$$

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The DOSM for systems of all identical particles

Methodology

Methodology [2]:

- To choose l_i and calculate λ,
- To solve the ET compact equations for a purely orbital excitation (to find $\tilde{\rho}_0$ and $\tilde{p}_0$),
- To compute ϕ,

$$\phi = \frac{\lambda}{N\tilde{p}_0\, T'(\tilde{p}_0)} \sqrt{\frac{k}{C_N^2 \mu}} \text{ with } \mu = \frac{\tilde{p}_0}{N T'(\tilde{p}_0)}$$

$$\text{and } k = \frac{2N\tilde{p}_0}{\tilde{\rho}_0{}^2} T'(\tilde{p}_0) + \frac{N\tilde{p}_0{}^2}{\tilde{\rho}_0{}^2} T''(\tilde{p}_0) + C_N^2 V''(\tilde{\rho}_0)$$

[2] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

Methodology

Methodology [2]:

- To choose l_i and calculate λ,
- To solve the ET compact equations for a purely orbital excitation (to find $\tilde{\rho}_0$ and $\tilde{p}_0$),
- To compute ϕ,
- To choose n_i and calculate Q (do not forget ϕ),

$$\boldsymbol{Q_\phi} = \phi \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right) + \lambda$$

[2] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

Methodology

Methodology [2]:

- To choose l_i and calculate λ,
- To solve the ET compact equations for a purely orbital excitation (to find $\tilde{\rho}_0$ and $\tilde{p}_0$),
- To compute ϕ,
- To choose n_i and calculate Q (do not forget ϕ),
- To resolve ET compact equations with this Q.

$$\begin{cases} E = NT(p_0) + C_N^2 V(\rho_0) \\ \sqrt{C_N^2}\rho_0 p_0 = Q_\phi \\ Np_0 T'(p_0) = C_N^2 \rho_0 V'(\rho_0) \end{cases}$$

[2] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

Example

Hamiltonian: $T(x) = Fx^\alpha$ $V(x) = \text{sgn}(\beta) G x^\beta$

$\Rightarrow$ Determination of $\tilde{\rho}_0$ and $\tilde{p}_0$:

$$\tilde{\rho}_0 = \left(\frac{N\alpha F \lambda^\alpha}{|\beta| G \sqrt{C_N}^{\alpha+2}} \right)^{1/(\alpha+\beta)} \quad \text{and} \quad \tilde{p}_0 = \frac{\lambda}{\sqrt{C_N}\tilde{\rho}_0}$$

Result is from [3] Chevalier et al. (2022) Few-Body Syst., **63**, 40

The DOSM for systems of all identical particles

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$$\phi = \sqrt{\alpha + \beta}$$

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$\Rightarrow$ Computation of ϕ:

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$\Rightarrow$ Final spectrum:

$$E = \text{sgn}(\beta)(\beta + \alpha) \left(\left(\frac{NF}{|\beta|} \right)^\beta \left(\frac{G}{\alpha} \right)^\alpha \left(\sqrt{C_N^2} \right)^{2\alpha - \alpha\beta} Q_\phi^{\alpha\beta} \right)^{1/(\alpha+\beta)}$$

$$\text{with } Q_\phi = \sqrt{\alpha + \beta} \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right) + \lambda$$

Result is from [3] Chevalier et al. (2022) Few-Body Syst., **63**, 40

The DOSM for systems of all identical particles

Tests

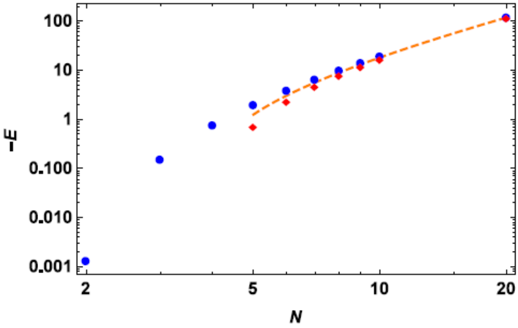

Figure: Biding energy for weakly-interacting bosons (gaussian interaction) with $d = 3$ - Exact results in circles, ET results in diamonds, $\phi = 1.82$ results in dashed line.

Results are from [4] Semay (2015) Few-Body Syst., **56**, 149

The DOSM for systems of all identical particles

Tests

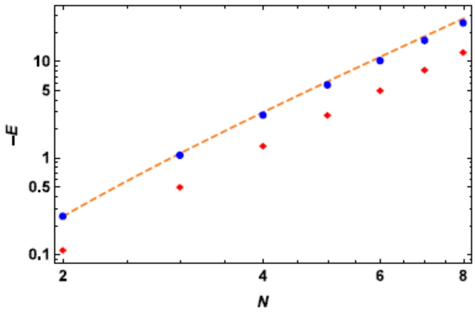

Figure: Biding energy for self-gravitating bosons (coulomb interaction) with $d = 3$ - Exact results in circles, ET results in diamonds, $\phi = 1$ results in dashed line.

Results are from [4] Semay (2015) Few-Body Syst., **56**, 149

The DOSM for systems of all identical particles

Tests

$n_1 + n_2$	$l_1 + l_2$	Exact	ET ($\phi = 2$)	ET ($\phi = \sqrt{2}$)
0	0	2.128	2.468	2.165
0	1	2.606	2.914	2.662
1	0	2.739	3.300	2.842
0	2	2.959	3.300	3.080
1	1	3.125	3.646	3.237
0	3	3.299	3.646	3.448
2	0	3.260	3.961	3.387
1	2	3.422	3.961	3.589
0	4	3.581	3.961	3.780
Δ			15%	3.8%

Table: Eigenmasses in GeV given by a model of light baryons ($D = 3$ and $N = 3$).

Results are from [4] Semay (2015) Few-Body Syst., **56**, 149

The DOSM for systems of all identical particles

Tests

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The DOSM for systems of $N_a + 1$ particles

Recap of the necessary informations about ET

Compact equations of ET ($N_a + 1$ particles): [5]

$$\begin{cases} E = N_a T_a(p'_a) + T_b(P_0) + C^2_{N_a} V_{aa}(r_{aa}) + N_a V_{ab}(r'_0) \\ N_a T'_a(p'_a)\frac{p_a^2}{p'_a} = C_{N_a} V'_{aa}(r_{aa}) r_{aa} + \frac{N_a-1}{2} V'_{ab}(r'_0)\frac{r_{aa}^2}{r'_0} \\ T'_b(P_0)P_0 + \frac{1}{N_a} T'_a(p'_a)\frac{P_0^2}{p'_a} = N_a V'_{ab}(r'_0)\frac{R_0^2}{r'_0} \\ p_a r_{aa}\sqrt{C^2_{N_a}} = Q(N_a) \\ P_0 R_0 = Q(2) \end{cases}$$

- with ${p'_a}^2 = p_a^2 + \frac{p_0^2}{N_a^2}$ and ${r'_0}^2 = \frac{N_a-1}{2N_a} r_{aa}^2 + R_0^2$

[5] Cimino, Semay (2022) Braz. J. Phys, **52**, 45

The ET and DOSM for systems of $N_a + 1$ particles

To apply the DOSM to the compact equations

Strategy [3]:

(1) To start with a classical purely orbital solution
(2) To start a radial perturbation (still classically)
(3) To quantify the perturbation

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[3] Chevalier, Willemyns, Cimino, Semay (2022) Few-Body Syst., **63**, 40

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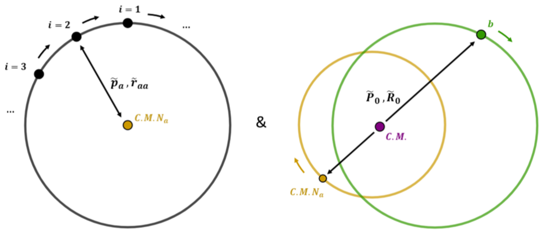

[3] Chevalier, Willemyns, Cimino, Semay (2022) Few-Body Syst., **63**, 40

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To apply the DOSM to the compact equations

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… [Insert a lot of Taylor developements] …

[3] Chevalier, Willemyns, Cimino, Semay (2022) Few-Body Syst., **63**, 40

The ET and DOSM for systems of $N_a + 1$ particles

To apply the DOSM to the compact equations

Strategy [3]:

(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

(3) To quantify the perturbation

After comparison of ET and DOSM, we get:

$$\phi_a = \frac{\lambda_a}{B_a}\sqrt{\frac{A''}{C^2_{N_a} m}} \quad \text{and} \quad \phi_b = \frac{\lambda_b}{B_b}\sqrt{\frac{B''}{m}}$$

where $B_a = T'_a(\tilde{p}'_a)\frac{N_a \tilde{p}^2_a}{\tilde{p}'_a}$ and $B_b = T'_a(\tilde{p}'_a)\frac{\tilde{P}^2_0}{N_a \tilde{p}'_a} + T'_b(\tilde{P}_0)\tilde{P}_0$,

$m = \sqrt{\mu_a \mu_b}$,

$$\begin{cases} A'' = \sqrt{\frac{\mu_b}{\mu_a}}k_a \\ B'' = \sqrt{\frac{\mu_a}{\mu_b}}k_b \end{cases} \quad \text{if } k_c = 0,$$

$$\begin{cases} A'' = \sqrt{\frac{\mu_b}{\mu_a}}k_a - \frac{k_c}{2} \\ B'' = \sqrt{\frac{\mu_b}{\mu_a}}k_a + \frac{k_c}{2} \end{cases} \quad \text{if } \epsilon = \frac{1}{k_c}\left(\sqrt{\frac{\mu_a}{\mu_b}}k_b - \sqrt{\frac{\mu_b}{\mu_a}}k_a\right) = 0,$$

$$\begin{cases} A'' = \sqrt{\frac{\mu_b}{\mu_a}}k_a - \frac{k_c}{2}\left(\frac{\epsilon}{|\epsilon|}\sqrt{1+\epsilon^2} - \epsilon\right) \\ B'' = \sqrt{\frac{\mu_a}{\mu_b}}k_b + \frac{k_c}{2}\left(\frac{\epsilon}{|\epsilon|}\sqrt{1+\epsilon^2} - \epsilon\right) \end{cases} \quad \text{if } \epsilon \neq 0,$$

$$\mu_a = \frac{\tilde{p}'_a}{N_a T'_a(\tilde{p}'_a)} \text{ and } \mu_b = \left(\frac{T'_a(\tilde{p}'_a)}{N_a \tilde{p}'_a} + \frac{T'_b(\tilde{P}_0)}{\tilde{P}_0}\right)^{-1},$$

$$k_a = \frac{N_a T''_a(\tilde{p}'_a)\tilde{p}^4_a}{\tilde{r}^2_{aa}\tilde{p}'^2_a} + \frac{N_a T'_a(\tilde{p}'_a)\tilde{p}^2_a}{\tilde{r}^2_{aa}}\left(\frac{3}{\tilde{p}'_a} - \frac{\tilde{p}^2_a}{\tilde{p}'^3_a}\right) + C^2_{N_a}V''_{aa}(\tilde{r}_{aa})$$
$$+ \frac{(N_a-1)^2\tilde{r}^2_{aa}}{4N_a\tilde{r}'^2_0}V''_{ab}(\tilde{r}'_0) + \frac{(N_a-1)}{2}\left(\frac{1}{\tilde{r}'_0} - \frac{(N_a-1)\tilde{r}^2_{aa}}{2N_a\tilde{r}'^3_0}\right)V'_{ab}(\tilde{r}'_0),$$

$$k_b = \frac{T''_a(\tilde{p}'_a)\tilde{P}^4_0}{N^3_a\tilde{R}^2_0\tilde{p}'^2_a} + \frac{T''_b(\tilde{P}_0)\tilde{P}^2_0}{\tilde{R}^2_0} + \frac{T'_a(\tilde{p}'_a)\tilde{P}^2_0}{N_a\tilde{R}^2_0}\left(\frac{3}{\tilde{p}'_a} - \frac{\tilde{P}^2_0}{N^2_a\tilde{p}'^3_a}\right)$$
$$+ \frac{2T'_b(\tilde{P}_0)\tilde{P}_0}{\tilde{R}^2_0} + \frac{N_a\tilde{R}^2_0}{\tilde{r}'^2_0}V''_{ab}(\tilde{r}'_0) + N_a\left(\frac{1}{\tilde{r}'_0} - \frac{\tilde{R}^2_0}{\tilde{r}'^3_0}\right)V'_{ab}(\tilde{r}'_0),$$

$$k_c = \frac{2\tilde{p}^2_a\tilde{P}^2_0}{N_a\tilde{p}'^2_a\tilde{r}_{aa}\tilde{R}_0}\left(T''_a(\tilde{p}'_a) - \frac{T'_a(\tilde{p}'_a)}{\tilde{p}'_a}\right) + \frac{(N_a-1)\tilde{r}_{aa}\tilde{R}_0}{\tilde{r}'^2_0}\left(V''_{ab}(\tilde{r}'_0) - \frac{V'_{ab}(\tilde{r}'_0)}{\tilde{r}'_0}\right).$$

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[3] Chevalier, Willemyns, Cimino, Semay (2022) Few-Body Syst., **63**, 40

The DOSM for systems of $N_a + 1$ particles

Tests

Test: $H = \sum_{i=1}^{3} |\vec{p}_i| + (\vec{r}_1 - \vec{r}_2)^2 + \kappa \sum_{i=1}^{2} (\vec{r}_i - \vec{r}_3)^2 \quad (D = 3)$

Result are from [3] Chevalier et al. (2022) Few-Body Syst., **63**, 40

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$$T_a(x) = T_b(x) = |x| \qquad V_{aa}(x) = x^2 \qquad V_{ab}(x) = \kappa x^2$$

Result are from [3] Chevalier et al. (2022) Few-Body Syst., **63**, 40

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- $\kappa = 0.1, 10$:

(n_a, n_b, l_a, l_b)	κ	Exact [8,9]	ET	$\Delta(\%)$	DOSM	$\Delta(\%)$
$(0,0,0,0)$	0.1	5.288	5.597	5.5	5.307	0.4
	10	14.506	15.352	5.8	14.699	1.3
$(0,0,1,1)$	0.1	7.515	7.868	4.7	7.625	1.5
	10	20.340	21.580	6.1	21.032	3.4
$(1,0,0,0)$	0.1	8.067	8.570	6.2	8.010	0.7
	10	19.134	20.272	5.9	19.291	0.8
$(0,1,0,0)$	0.1	6.750	6.970	3.2	6.571	2.7
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Result are from [3] Chevalier et al. (2022) Few-Body Syst., **63**, 40

The DOSM for systems of $N_a + 1$ particles

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[11] Semay, Sicorello (2018) Few-Body Syst., **59**, 119

[10] Semay, Cimino, Willemyns (2020) Few-Body Syst., **61**, 19

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$\Rightarrow$ Many applications: hadronic, nuclear, atomic and molecular, solid state physics...

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